On 2-blocks of finite groups with elementary abelian defect groups of order 8

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Introduction

k: alg. closed field of characteristic $p > 0$, G : finite group.
The group algebra kG decomposes uniquely into a product of indecomposable factors $kG = \prod_i B_i$; the B_i are the *blocks of* kG.

• If P is a Sylow p-subgroup of G, then for any block B of kG, the map

$$B \otimes_{kP} B \to B,\ x \otimes y \to xy,\ x, y \in B$$

splits as map of (B, B)-bimodules:

$$z \to \frac{1}{|G : P|} \sum_{g \in G/P} zg \otimes g^{-1},\ z \in B \text{ is a splitting.}$$

A *defect group* of B is a p-subgroup P of G minimal such that $B \otimes_{kP} B \to B$ splits as map of (B, B)-bimodules.

• The defect groups of B are unique upto conjugation in G.

• B has trivial defect group if and only if B is a matrix algebra over k.

• kG has blocks with non-trivial defect groups if and only if $p||G|$.

Example 1. $G = S_3$

- If $p \geq 5$ (or $p = 0$),

$$kS_3 = k \times k \times \mathrm{Mat}_2(k);$$

  all defect groups are trivial

- If $p = 2$,

$$kS_3 = \mathrm{Mat}_2(k) \times kC_2;$$

  defect groups are $\{1\}$ and C_2 respectively

- If $p = 3$, kS_3 is a block with defect group C_3.

Example 2. $G = 2M_{22}$, $p = 3$.
$|G| = 2^8 \times 3^2 \times 5 \times 7 \times 11 = 887040$.
Sylow 3-subgroups of G are elementary abelian, so possible defect groups are 1, C_3, $C_3 \times C_3$.
There are 9 blocks:

B	$\dim_k B$	defect group
B_1	2025	1
B_2	2025	1
B_3	9801	1
B_4	15786	1
B_5	15786	1
B_6	97902	C_3
B_7	167400	C_3
B_8	244368	$C_3 \times C_3$
B_9	331767	$C_3 \times C_3$

B : block of kG, P : defect group of B.

Problem : Describe the module category $\mathrm{mod}(B)$of B of kG in terms of "local structure" , i.e., $P + \cdots$.

Conjecture. (Donovan '87)

For a fixed P, there are only finitely many possibilites for $\mathrm{mod}(B)$.

- If $P = 1$, then $B = Mat_n(k)$, hence $\mathrm{mod}(B) \sim \mathrm{mod}(k)$.

Example 2. $G = 2M_{22}$, $p = 3$; 4 blocks with non-trivial defect groups:

B	B_6	B_7	B_8	B_9
$\dim_k B$	97902	167400	244368	331767
P	C_3	C_3	$C_3 \times C_3$	$C_3 \times C_3$

- $\mathrm{mod}(B_6) \sim \mathrm{mod}(B_7) \sim \mathrm{mod}(kP \rtimes C_2)$ (Brauer-Dade cyclic block theory '66)
- $\mathrm{mod}(B_8) \sim \mathrm{mod}(kP \rtimes Q_8)$ (Danz-Külshammer, 2008)
- The derived bounded module category $D^b\mathrm{mod}(B_9) \sim D^b\mathrm{mod}(kP \rtimes Q_8)$ (Okuyama, '98)

B : block of kG, P : defect group of B, $Z(B)$: the center of B

ζ_B : $\dim_k Z(B)$, ℓ_B: number of isomorphism classes of simple B-modules.

- ℓ_B, $Z(B)$ and ζ_B are invariants of $\mathrm{mod}(B)$ and of $D^b\mathrm{mod}(B)$.
- ζ_B is the number of ordinary irreducible characters of G lying in the block B.
- $\ell_B \leq \zeta_B \leq \frac{1}{4}|P|^2 + 1$ (Brauer-Feit, '56).

$B:$ block of kG, $P:$ defect group of B

- $P = 1$, $\ell_B = \zeta_B = 1$.
- P cyclic, $\ell_B = e$, $\zeta_B = e + \frac{|P|-1}{e}$, some $e|p-1$ (Brauer, Dade, '66).
- $p = 2$, P Klein 4-group, dihedral, semi-dihedral or generalised quaternion, $\ell_B = 1$ or 3, $\exists$ formulae for ζ_B (Brauer, Olsson '74).
- P admits only the trivial fusion system (e.g. $P = C_4 \times C_2$), $\ell_B = 1$, $\zeta_B = \zeta_{kP}$ (Broué-Puig, '80).

Theorem (K-Koshitani-Linckelmann, 2010)

If $P = C_2 \times C_2 \times C_2$, then $\zeta_B = 8$ and $\ell_B = 1, 3, 4$ or 7.

What took so long? CFSG + modular analogues of Deligne-Lusztig theory.

Local-global conjectures

Brauer's first main theorem

There is a bijection $B \leftrightarrow \tilde{B}$ between the set of blocks B of kG with P as a defect group and the set of blocks $\tilde{B}$ of $kN_G(P)$ with P as a defect group.

The bijection can be explicitly described: Write

$$1_B = \sum_{g \in G} \alpha_g g, \ \alpha_g \in k.$$

Then,

$$1_{\tilde{B}} = \sum_{g \in C_G(P)} \alpha_g g.$$

$\tilde{B}$ is the *Brauer correspondent* of B.

B : block of kG, $\tilde{B}$: Brauer correspondent of B, a block of $kN_G(P)$

Theorem (Kulshammer)

$\mathrm{mod}(\tilde{B}) \sim \mathrm{mod}(k_\alpha P \rtimes E)$ for a p'-subgroup $E \leq N_G(P)/C_G(P) \leq \mathrm{Aut}(P)$ and an element $\alpha \in H^2(E, k^*)$.

The group E is the *inertial quotient* of B.

Weight Conjecture, abelian case (Alperin, '87)

Suppose P abelian. Then $\ell_B = \ell_{\tilde{B}}$.

(General version of Alperin's weight conjecture involves Brauer correspondents of B in groups $N_G(Q)$ for proper subgroups Q of P.)

The above conjecture (in the abelian case) is implied by:

Abelian Defect Group Conjecture (Broué, '88)

Suppose P abelian. Then $D^b\mathrm{mod}(B) \sim D^b\mathrm{mod}(\tilde{B})$.

Example 2. $G = 2M_{22}$, $char(k) = 3$; 4 blocks with non-trivial defect groups:

B	B_6	B_7	B_8	B_9
P	C_3	C_3	$C_3 \times C_3$	$C_3 \times C_3$
E	C_2	C_2	Q_8	Q_8
α	1	1	1	1

- $\mathrm{mod}(B_6) \sim \mathrm{mod}(B_7) \sim \mathrm{mod}(kP \rtimes C_2)$ (Brauer-Dade cyclic block theory '66)
- $\mathrm{mod}(B_8) \sim \mathrm{mod}(kP \rtimes Q_8)$ (Danz-Külshammer, 2008)
- The derived bounded module category $D^b\mathrm{mod}(B_9) \sim D^b\mathrm{mod}(kP \rtimes Q_8)$ (Okuyama, '98)

So, abelian defect group conjecture and weight conjecture hold for $2M_{22}$, $p = 3$.

Abelian defect group conjecture known to hold if P cyclic, $P = C_2 \times C_2$ (Rickard '86), or if P admits only trivial fusion system (Puig '80).
In addition, weight conjecture known to hold if P is dihedral, semi-dihedal or generalised quaternion (Brauer-Olsson '74).

Theorem (K-Koshitani-Linckelmann, 2010)

If $P = C_2 \times C_2 \times C_2$, then $\zeta_B = 8$ and $\ell_B = 1, 3, 4$ or 7.

is really

Theorem'

If $P = C_2 \times C_2 \times C_2$, then B and $\tilde{B}$ are isotypic. In particular, $Z(B) \cong Z(\tilde{B})$ and $\ell_B = \ell_{\tilde{B}}$.

- First case of weight conjecture for all blocks with a given defect group since conjecture was announced.
- Only known case of weight conjecture where the defect group is of wild representation type and admits non-trivial fusion.
- Abelian defect group conjecture still open for $P = C_2 \times C_2 \times C_2$.

Idea of proof

$P = C_2 \times C_2 \times C_2$, $\tilde{B}$: Brauer correspondent of B, E : inertial quotient of B, p'-subgroup of $\mathrm{Aut}(P)$.

(I) Theorem' is equivalent to Theorem.

- (Rouquier '97) There is a stable equivalence of Morita type between B and $\tilde{B}$.

(II) Theorem' true if true for blocks of quasi-simple groups.

- (Landrock , '81) Theorem' true unless $E \cong C_7$ or $E \cong F_{21}$ and $\zeta_B = 5, 7$.

(III) Theorem' true for blocks of quasi-simple groups.

- There are "very few" non-nilpotent blocks of kG, G a quasi-simple group which have for defect group an elementary abelian 2-group of rank ≥ 3. (Uses Fong-Srinivasan, Broué-Malle-Michel, Cabanes-Enguehard, Bonnafé-Rouquier, Geck-Hiss)